

# Quantum Mechanics and Mathematical Economics are Isomorphic

John von Neumann between Physics and Economics<sup>1</sup>

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## **Abstract**

A methodological discussion is proposed, aiming at showing that game theory in particular (and mathematical economics in general) and quantum mechanics are isomorphic. This result relies on the equivalence of the two fundamental operators employed in the two fields, namely, the expected value in economics and the density matrix in quantum physics. This coincidence can be traced back to the contributions of von Neumann in both disciplines.

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# 1 Introduction

Over the last fifteen years, a growing amount of attention has been devoted to the history of game theory. Among other reasons, this interest can largely be justified on the basis of the Nobel prize to Nash, Harsanyi and Selten in 1994.<sup>1</sup>

However, the literature dealing with the history of game theory mainly adopts an *inner* perspective, i.e., an angle that allows us to reconstruct the developments of this sub-discipline under the general headings of economics. My aim is different, to the extent that I intend to propose an interpretation of the formal relationships between game theory (and economics) and the hard sciences.

A largely accepted view maintains that the foundations of both game theory and, in general, contemporary mathematical economics are to be found in *The Theory of Games and Economic Behavior* (von Neumann and Morgenstern, 1944).<sup>2</sup> From that work onwards, we observe the widespread use of the operator known as *expected value* in economics. Traditionally, and correctly, the roots of such operator are traced back to statistics (see Fishburn and Wakker, 1995, and the references therein). However, this view

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<sup>1</sup>On the history of game theory, see Aumann (1985, 1987, 1999), Weintraub (1992), Leonard (1994, 1995) and Myerson (1999), *inter alia*. On the appraisal of the role of game theory in contemporary economics, see van Damme and Weibull (1995).

<sup>2</sup>In von Neumann and Morgenstern (1944), the concept of stable set for cooperative games complements the minimax theorem (von Neumann, 1928) for noncooperative games. However, the latter is applicable only in zero sum games. Nash (1950) introduced an equilibrium concept for non-zero sum noncooperative games, and the bargaining solution for cooperative games (Nash, 1951).

does not grasp an interesting aspect in common between mathematical economics and physics, in particular quantum physics, that is, the fact that both theories transmit a largely analogous message concerning our knowledge (or representation) of real world phenomena. Our *ex ante* knowledge (i.e., our forecasting capability) is defined in the same probabilistic terms, independently of whether we examine the behaviour of subatomic particles or the behaviour of firms and consumers.

What I want to show in the remainder is that this coincidence (pointing ultimately to the idea that our knowledge is modelled in the same way irrespectively of the specific field<sup>3</sup>), has been largely shaped through the activity of a single researcher, John von Neumann, who, over a dozen years (1932-44) has ‘fixed’ the basic ideas in both disciplines by giving them the same methodology, thus making them largely *isomorphic* to each other. The key to this interpretation is the operator labelled as *expected value* by economists and *density matrix* by physicists.

In the first edition (in German) of *Mathematical Foundations of Quantum Mechanics*, von Neumann (1932) adopts the term *Erwartungswert*, which in English is *expectation value*. This also appears in von Neumann and Morgenstern (1944), while in the following economic literature it usually becomes *expected value*. In physics, *expectation value* is used to indicate the elementary notion of the mathematical expectation of a random variable, while *density matrix* indicates, rather loosely speaking, the random outcome of an experiment.

In the following section, I offer a succinct view of the formal equivalence

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<sup>3</sup>On this issue, see Penrose (1989).

between the two theories (and the way they build up their predictive power), through a well known parable in quantum physics.

## 2 The isomorphism between mathematical economics and quantum physics: a sketch

Quantum physics (and its proper subset, quantum mechanics) can approximately be considered (not all physicists agree on this matter) as a generalisation of Newtonian physics (and mechanics). In particular, when the phenomena being investigated involve (i) speeds which are considerably lower than the speed of light in vacuum; (ii) sufficiently low gravity; and (iii) objects which are neither too small nor too big, then one can proceed according to the standard Newtonian model. Yet, when it comes either to the very basic particles (e.g., sub-atomic particles like electrons), or to the behaviour of objects characterised by an extremely high gravitational force (e.g., black holes, quasars and pulsars), quantal features become so relevant that they cannot be disregarded.

As an illustrative example, consider the story traditionally known as *Schrödinger's paradox* or *Schrödinger's cat-in-the-box*, which I am about to tell according to a *vulgata* commonly adopted in the current literature in the field.<sup>4</sup>

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<sup>4</sup>Erwin Schrödinger, together with Bohr, Einstein, Dirac, Heisenberg and Pauli, is one of the founding fathers of quantum physics. The remainder of this section borrows from Hawking and Penrose (1996) and Penrose (1997). For further (and much more technical) readings, see Sakurai (1985), Schwabl (1992), and Gasiorowicz (1996), *inter alia*.

This paradox refers to an experiment, where a cat is locked in a box, in front of a handgun, which is loaded.<sup>5</sup> The box is opaque to both light and sound, so that the physicist must open it to observe the cat's conditions. The trigger is linked to a sensor, located outside the box, in front of a bulb. Assume the physicist can switch on the bulb, producing a single quantum of light (a photon) at a time. The sensor measures the spin of the photon. In the remainder, I will assume conventionally the following. If the photon rotates rightwards (i.e., the spin is positive), then the sensor pulls the rope (and the trigger), and the handgun shoots the cat dead. If instead the photon rotates leftwards (i.e., the spin is negative), then the sensor does not pull the rope (and the trigger), so that the cat survives.

The paradox takes place in the physicist's mind before the experiment is carried out, or, equivalently, before he opens the box to see whether the cat is still alive or not. The physicist knows that, *ex post*, the cat is going to be either dead or alive. However, *ex ante*, the two states of the system-cat (i.e., *alive* and *dead*) as well as the two states of the world (that can be labelled as *the handgun shot because the spin was positive*, and *the handgun didn't shoot because the spin was negative*) coexist in the experimenter's mind. This paradox persists until he opens the box to observe the cat's

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<sup>5</sup>This can be viewed as a *Gedankenexperiment* (thought experiment), in that we must not necessarily lock a cat into a box to verify the validity of what follows. In general, a *Gedankenexperiment* "is consistent with the known laws of physics, even though it may not be technically feasible. So, measuring the acceleration due to gravity on the surface of the sun is a *Gedankenexperiment*, whereas measuring the Doppler shift of sunlight as seen from a space ship moving with twice the velocity of light is nonsense" (Gasiorowicz, 1996, p. 21, fn. 14). In the present case, replace *technically feasible* with *politically correct*.

health conditions.

I will formalise the experiment by introducing the following:

**Definition 1**  $\psi \equiv \{ca, cd\}$  is the wave function, or the state vector of the system (the cat).

In Definition 1, intuitively,  $ca$  = “the cat is alive”;  $cd$  = “the cat is dead”. Moreover,

**Definition 2**  $\vartheta \equiv \{hs, hds\}$  is the vector of the states of the world.

In Definition 2, intuitively,  $hs$  = “the handgun shot (because the spin was positive)”;  $hds$  = “the handgun did not shoot (because the spin was negative)”.

Finally, we have

**Definition 3**  $\varphi \equiv \{ca, cd\}$  is the state vector of the physicist carrying out the experiment.

The reason why  $\varphi$  seems to coincide with the wave function  $\psi$  will become clear in the remainder. The *ex ante* overlapping between the states of the system, is accounted for by the *total state vector of the system*:

$$|\psi_{tot}\rangle = \omega|ca\rangle + \rho|cd\rangle \quad (1)$$

where both  $\omega$  and  $\rho$  belong to  $\mathbb{C}$  and  $|\cdot\rangle$  is called *ket*. Its complement  $\langle\cdot|$  is called *bra*, and  $\langle\cdot|\cdot\rangle$  is called *Dirac's brackets*, after P.A.M. Dirac who introduced this terminology in 1930 (see Dirac, 1930, 1958<sup>4</sup>; see also Dirac, 1925).

As soon as the experimenter opens the box to observe the state of the system, the wave function  $\psi$  *collapses*, that is, there takes place the so-called *reduction of the state vector*  $|\psi_{tot}\rangle$ . Hence, the experimenter observes either  $ca$  or  $cd$ , but surely not both states at the same time. On the contrary, as long as the experimenter does not open the box, the two state *coexist in a quantal sense*.<sup>6</sup>

For reasons that I do not dwell upon here,  $\omega$  and  $\rho$  are complex numbers.<sup>7</sup> In particular, what I am interested in, is the information that the modulus of both  $\omega$  and  $\rho$  is equal to  $1/\sqrt{2}$ . To the square, this yields  $1/2$ , which is the probability that the photon have either a positive or a negative spin. This, as we shall see in the remainder, has relevant bearings upon my aim of showing the existence of an isomorphism between quantum physics and mathematical economics (game theory in particular).

The total state vector of the experimenter can be constructed to look like that of the system (but recall that the two vectors *do not coincide*):

$$\langle\varphi_{tot}| = \omega\langle ca| + \rho\langle cd|, \quad \omega = \rho = 1/\sqrt{2}. \quad (2)$$

In probabilistic terms, the experimenter's *ex ante* knowledge about the states that the system can take *ex post* is summarised by the so-called *density matrix*:

$$D = \frac{1}{2}|ca\rangle\langle ca| + \frac{1}{2}|cd\rangle\langle cd|, \quad (3)$$

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<sup>6</sup>Physicists have lively arguments concerning the alternative interpretation of such a statement, i.e., whether states  $ca$  and  $cd$  coexist in parallel universes or in the experimenter's mind. As far as the present paper is concerned this distinction, although intriguing, can be disregarded.

<sup>7</sup>See the appendix. For further discussion, see, *inter alia*, Sakurai (1985), Schwabl (1992) and Gasiorowicz (1996).



which is defined as the scalar product of the two total state vector, i.e.,  $|\psi_{tot}\rangle \cdot \langle\varphi_{tot}|$ . The introduction of the density operator can be traced back to Dirac (1930, 1958<sup>4</sup>) and reappears in von Neumann (1932 [1955]). This indicates that the isomorphism between (i) the theory of expected utility, game theory and contemporary mathematical economics in general, on one side; and (ii) quantum mechanics on the other side, can hardly be considered as accidental.

In *Theory of Games and Economic Behavior*, von Neumann makes use of a toolkit borrowed from Dirac's formalisation of quantum mechanics,<sup>8</sup> However, von Neumann does not bother to make it explicitly known either to economists or to physicists.

Expression (3) can be rewritten equivalently as follows:

$$D = \frac{1}{2}\langle ca|ca\rangle + \frac{1}{2}\langle cd|cd\rangle , \quad (4)$$

from which there emerges a clear analogy with what economists would call *expected value*. Indeed,  $\omega^2 = \rho^2 = 1/2$  are the probabilities that the spin of the photon be either positive or negative. Therefore, the density matrix corresponds to the expected value of the experiment, which an economist would write:

$$E(\text{experiment}) = \frac{1}{2}(ca | ca) + \frac{1}{2}(cd | cd) . \quad (5)$$

Notice that the above reads the same in both disciplines, independently of whether we write it as in (4) or as in (5), i.e., “with probability 1/2 I will

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<sup>8</sup>von Neumann himself motivates his interest in quantum physics on the basis of his dissatisfaction with Dirac's formalisation of the theory (Dirac, 1930, 1958<sup>4</sup>), in particular with the  $\delta$  function (see the author's preface in von Neumann, 1932 [1955]).

observe a dead cat because the cat is in fact dead (or, because the handgun shot, as the spin was positive), and with the same probability I will observe that the cat is still alive (because the handgun did not shoot, as the spin was negative)". Observe that the meaning of symbol  $|$  in brackets is exactly the same in both cases.

The above describes the procedure usually adopted in economics to solve games in mixed strategies.<sup>9</sup>

At this stage, one could rewrite (5) in terms of the vector of the states of the world,  $\vartheta$  :

$$E(esp) = \frac{1}{2}(ca | hds) + \frac{1}{2}(cd | hs) . \quad (6)$$

Now I would like to stress that, when economists build up the expected value of an agent's payoff, they face exactly the same kind of paradox which I have illustrated above concerning the reduction of the state vector in the *Schrödinger's cat* example. Hence, also in economics the issue is the transition from the *ex ante* (quantal) prevision to the *ex post* (Newtonian) observation. In both settings, the limit to our knowledge is inherently given by the **indeterminacy**<sup>10</sup> associated to (i) the measurement in physics and (ii)

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<sup>9</sup>The same discussions taking place amongst economists concerning the interpretation of a Nash equilibrium in mixed strategies as opposed to the pure-strategy equilibrium, also occur amongst physicists as to the credibility of the quantistic description of the world, as opposed to the Newtonian description. The latter discussion can be traced back to the famous dispute between Niels Bohr and Albert Einstein, who, while being one the fathers (although less than voluntarily) of quantum theory, claimed that "God doesn't toss any dies". For a contemporary revisitation of the dispute, see Hawking and Penrose (1996), with Hawking in Bohr's role and Penrose in Einstein's.

<sup>10</sup>This is the so-called Heisenberg's principle, which can be given a *vulgata* through the following example. We cannot know at the same time how fast an electron is moving, *and*

the (random) transition from a mixed strategy to a particular pure strategy in game theory.<sup>11</sup>

The analogy with well known situations in game theory is quite immediate. Consider the following game in normal form.

		$B$	
		$s$	$d$
$A$	$a$	$\pi_A(a, s), \pi_B(s, a)$	$\pi_A(a, d), \pi_B(d, a)$
	$b$	$\pi_A(b, s), \pi_B(s, b)$	$\pi_A(b, d), \pi_B(d, b)$

Matrix 1

In general, the game represented in matrix 1 has at least an equilibrium in mixed strategies (and possibly several in pure strategies). Define as  $\alpha$  the probability that player  $A$  chooses strategy  $a$ , and  $\beta$  the probability that player  $B$  chooses  $s$ . Obviously, the probabilities that  $b$  and  $d$  are chosen are, respectively,  $1 - \alpha$  and  $1 - \beta$ . We can write the expected value of the payoff accruing to player  $A$  :

$$E(\pi_A) = \alpha\beta\pi_A(a, s) + \alpha(1 - \beta)\pi_A(a, d) + (1 - \alpha)\beta\pi_A(b, s) + (1 - \alpha)(1 - \beta)\pi_A(b, d) , \quad (7)$$

which  $A$  must maximise w.r.t.  $\alpha$ . We can construct  $E(\pi_A)$  likewise. In the physicists' jargon, expression (7) could be defined as the *density matrix of* where it is. The knowledge we can produce confines to the wave function, yielding the probability that the electron be within a certain interval, given its speed.

<sup>11</sup>Observe that the interpretation I am proposing here does not modify the established wisdom about the source and evolution of the theory of expected utility (see, *inter alia*, Fishburn and Wakker, 1995).

the game for player  $A$ , and could be rearranged as follows:

$$\begin{aligned} D(\pi_A) = & \alpha\beta\langle\pi_A(a, s)|s\rangle + \alpha(1-\beta)\langle\pi_A(a, d)|d\rangle + \\ & (1-\alpha)\beta\langle\pi_A(b, s)|s\rangle + (1-\alpha)(1-\beta)\langle\pi_A(b, d)|d\rangle. \end{aligned} \quad (8)$$

One could raise the objection that, in a game, probabilities are chosen by players, while in physics they are given by Nature. As a counter-objection, one could put forward the model of the market for lemons (Akerlof, 1970) or the games illustrated in Bassan, Scarsini and Zamir (1997), or else point to the games with incomplete information (Harsanyi, 1967), where the states of the world are mutually exclusive and realise with probabilities dictated by Nature.

As an illustrative example of the latter class of games, one could examine the behaviour of a firm (say, for simplicity, a monopolist) in a market where the demand function is stochastic (see Leland, 1972; Klemperer and Meyer, 1986, 1989, *inter alia*). Suppose demand can take any value  $\vartheta_i$ ,  $i = 1, 2, \dots, n$ , with  $\vartheta_i > \vartheta_{i-1}$ . Define as

- $p_i(\vartheta_i)$  the probability of  $\vartheta_i$ , with  $\sum_{i=1}^n p_i(\vartheta_i) = 1$ ;
- $\pi_i(\vartheta_i, q_i)$  the profit accruing to the firm in state  $i$ .

The states of the world  $\{\vartheta_i\}$  may refer to the political situation, the oil crisis, and so on and so forth. What matters is that they relate the profits of the firm to exogenous environmental features controlled by Nature. *Ex ante*, the firm chooses the output level in order to maximise the *expected value* of profits, i.e.,  $E(\pi) = \sum_{i=1}^n p_i(\vartheta_i) \cdot \pi_i(\vartheta_i, q_i)$ . This could be translated in the jargon of physics by saying that the firm aims at maximising the density

matrix of profits. *Ex post*, the equilibrium profits accruing to the firm will depend upon the precise realisation of  $\vartheta$ . That is, the value of  $\pi$  becomes known as soon as the reduction of the state vector takes place.

There are economic problems where the economic agent's *ex ante* knowledge is sufficient to reveal with certainty how the situation will look like *ex post*. To illustrate this issue, consider the following simplified version of Akerlof's (1970) lemons market, where lemons are low-quality second-hand cars. A customer, interested in buying a second-hand car, faces a population of sellers, each offering a car whose quality can be either  $q = H$  or  $q = L$  ( $H > L$ ), with probability  $p(H) = p(L) = 1/2$ . The metaphor usually adopted to tell this story is that Nature extracts randomly one seller from the population of sellers, half of which have good quality cars. Therefore, *a priori*, the potential buyer can compute the expected quality as  $E(q) = (H + L)/2$ . This is also the price the buyer is willing to pay. In the symbology adopted in physics, such expression would write as the following density matrix of quality:

$$D(q) = \frac{1}{2}\langle H|H\rangle + \frac{1}{2}\langle L|L\rangle . \quad (9)$$

This, *ex ante*, entails the overlapping of qualities (or states) in a quantal sense. In the present case, however, the collapse of this overlapping takes place before the transaction is carried out. To see this, examine the sellers' perspective. Each seller knows the quality of his own car. Hence, any car which is being offered on the market, given the buyer's evaluation  $E(q) = (H + L)/2$ , must necessarily be a poor quality one. Therefore market fails because the buyer know that he could only buy a lemon at too high a price.

## Appendix: Dirac's notation

Dirac (1930, 1958<sup>4</sup>) introduced the following notation that applies to finite dimensional vector spaces as well as to Hilbert spaces.

Associate to the wave function  $\psi$  a state vector  $|\psi\rangle$ , called *ket*, and define the quantity  $\langle\varphi|$  as *bra*.<sup>12</sup> Then,

$$\int dx \varphi^* \psi = \langle\varphi|\psi\rangle \quad (\text{a1})$$

The integral involving an operator  $W$  can be written as

$$\int dx \varphi^* W \psi = \langle\varphi|W\psi\rangle = \langle\varphi|W|\psi\rangle \quad (\text{a2})$$

If a number  $w$  is considered, then it can be taken out of brackets, with:

$$\langle\varphi|w\psi\rangle = w \langle\varphi|\psi\rangle \quad (\text{a3})$$

and

$$\langle w\varphi|\psi\rangle = w^* \langle\varphi|\psi\rangle \quad (\text{a4})$$

The density operator is

$$D = |\psi\rangle \langle\psi| \quad (\text{a5})$$

Now, since  $|\psi\rangle = \sum_i \omega_i |u_i\rangle$ , we have

$$D = |\psi\rangle \langle\psi| = \sum_i (\omega_i)^2 |u_i\rangle \langle u_i| \quad (\text{a6})$$

or, more generally,

$$D = |\psi\rangle \langle\psi| = \sum_i \omega_i \omega_j |u_i\rangle \langle u_j| \quad (\text{a7})$$

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<sup>12</sup> $\varphi^*$  is the complex conjugate wave function. See Gasiorowicz (1996).

## References

- [1] Akerlof, G. (1970), “The Market for Lemons: Quality Uncertainty and the Market Mechanism”, *Quarterly Journal of Economics*, **84**, 488-500.
- [2] Aumann, R.J. (1985), “What Is Game Theory Trying to Accomplish?”, in Arrow, K. and S. Honkapohja (eds.), *Frontiers in Economics*, Oxford, Blackwell.
- [3] Aumann, R.J. (1987), “Game Theory”, in Eatwell, J., M. Milgate and P. Newman (eds.), *The New Palgrave: A Dictionary of Economics*, London, Macmillan.
- [4] Aumann, R.J. (1999), “Game Theory in Israel: Looking Backward and Forward”, paper presented at ASSET99, Tel Aviv, October 10-11, 1999.
- [5] Bassan, B., M. Scarsini and S. Zamir (1997), “I Don’t Want to Know! Can It Be Rational?” discussion paper n. 158, Center for Rationality and Interactive Decision Theory, The Hebrew University of Jerusalem.
- [6] Dirac, P.A.M. (1925), “The Fundamental Equations of Quantum Mechanics”, *Proc. Roy. Soc. (London) A*, **109**, 642-653.
- [7] Dirac, P.A.M. (1930, 1958<sup>4</sup>), *The Principles of Quantum Mechanics*, Oxford, Oxford University Press.
- [8] Fishburn, P. and P. Wakker (1995), “The Invention of the Independence Condition for Preferences”, *Management Science*, **41**, 1130-1144.
- [9] Gasiorowicz, S. (1996), *Quantum Physics. Second Edition*, New York, Wiley.

- [10] Hawking, S. and R. Penrose (1996), *The Nature of Space and Time*, Princeton, NJ, Princeton University Press.
- [11] Klemperer, P. and Meyer, M. (1986), “Price Competition vs Quantity Competition: The Role of Uncertainty”, *RAND Journal of Economics*, **17**, 618-38.
- [12] Klemperer, P. and Meyer, M. (1989), “Supply Function Equilibria in Oligopoly under Uncertainty”, *Econometrica*, **57**, 1243-77.
- [13] Leland, H. (1972), “Theory of the Firm Facing Uncertain Demand”, *American Economic Review*, **62**, 278-91.
- [14] Leonard, R.J. (1994), “Reading Cournot, Reading Nash: The Creation and Stabilisation of the Nash Equilibrium”, *Economic Journal*, **104**, 492-511.
- [15] Leonard, R.J. (1995), “From Parlor Games to Social Science: von Neumann, Morgenstern, and the Creation of Game Theory”, *Journal of Economic Literature*, **33**, 730-761.
- [16] Myerson, R. (1999), “Nash Equilibrium and the History of Economic Theory”, *Journal of Economic Literature*, **37**, 1067-1082.
- [17] Nash, J.F. (1950), “Equilibrium Points in n-Person Games”, *Proceedings of the National Academy of Science of the USA*, **36**, 48-49.
- [18] Nash, J.F. (1951), “Non-Cooperative Games”, *Annals of Mathematics*, **54**, 289-295.



- [19] Penrose, R. (1989), *The Emperor's New Mind*, Oxford, Oxford University Press.
- [20] Penrose, R. (1997), *The Large, the Small and the Human Mind*, Cambridge, Cambridge University Press.
- [21] Sakurai, J.J. (1985), *Modern Quantum Mechanics. Revised Edition*, New York, Addison-Wesley.
- [22] Schwabl, F. (1992), *Quantum Mechanics*, Berlin, Heidelberg, Springer-Verlag.
- [23] van Damme, E. and J. Weibull (1995), "Equilibrium in Strategic Interaction: The Contributions of John C. Harsanyi, John F. Nash and Reinhard Selten", *Scandinavian Journal of Economics*, **97**, 15-40.
- [24] von Neumann, J. (1928), "Zur Theorie der Gesellschaftsspiele", *Mathematische Annalen*, **100**, 295-320.
- [25] von Neumann, J. (1932), *Mathematische Grundlagen der Quantenmechanik*, Springer Verlag, Berlin-Heidelberg (*Mathematical Foundations of Quantum Mechanics*, Princeton, NJ, Princeton University Press, 1955).
- [26] von Neumann, J. e O. Morgenstern (1944), *The Theory of Games and Economic Behavior*, Princeton, NJ, Princeton University Press.
- [27] Weintraub, R. (1992, ed.), *Toward a History of Game Theory*, in *History of Political Economy*, **24** (Supplement).